Towards a Noether–like conservation law theorem for one dimensional reversible cellular automata [1]

Tim Boykett

Time's Up Research Department
and
Department of Mathematics, Johannes–Kepler University, Linz, Austria

Abstract

Evidence and results suggesting that a Noether–like theorem for conservation laws in 1D RCA can be obtained. Unlike Noether's theorem, the connection here is to the maximal congruences rather than the automorphisms of the local dynamics.

We take the results of Takesue and Hattori (1992) on the space of additive conservation laws in one dimensional cellular automata. In reversible automata, we show that conservation laws correspond to the null spaces of certain well-structured matrices.

It is shown that a class of conservation laws exist that correspond to the maximal congruences of index 2. In all examples investigated, this is all the conservation laws. Thus we conjecture that there is an equality here, corresponding to a Noether–like theorem.

Key words: reversible cellular automata, conservation law, algebraic structure, maximal congruences
PACS: 02.10.ox, 89.75.Fb, 05.45.-a
1991 MSC: 68Q80, 37B15, 70H33, 68Q10, 05E99, 08A99

1 Introduction

This paper outlines some investigations into the conservation rules of one dimensional reversible cellular automata (RCA). Using results from Takesue

---

and Hattori, we obtained examples for all RCA of low order (≤ 9). Based upon these observations, we noted a strong connection with congruences, in particular maximal congruences. We propose a conjecture relating the two, akin to Noether's result about conservation laws in continuous systems.

Conservation laws in physical systems are of great help in interpreting the properties of these systems. The investigation of conservation laws in models of physical systems, i.e. cellular automata, seems thus relevant. In particular, (microscopic) reversibility in physical systems is often of importance, leading us to investigate reversible cellular automata.

2   Reversible cellular automata

One dimensional reversible cellular automata (1DRCA or simply RCA) are invertible, continuous mappings of A^Z to itself that commute with the shift operator. A is a finite set of cell states. Z can be either the integers or the integers modulo n for some n. The metric measures the size of the matching middle, i.e. if $a_i = b_i$ for all $|i| < k$ and $a_i \neq b_i$ for some $|i| = k$ then $d(a, b) = \frac{1}{2^k}$. The shift operator σ is defined by $\sigma(a)_i = a_{i+1}$. CA can be defined by the set A and the *local rule* $f : A^n \rightarrow A$ [Ric72]. Our CA are binary (radius one half), the local rule is a binary operation. If f is the local rule, then F the global mapping, is defined by $F(a)_i = f(a_i, a_{i+1})$.

We need only consider a class of $(2, 2)$–algebras known as *semicentral bigroupoids* to investigate RCA [Ped92,Boy03]. These are defined as $(A, \bullet, \circ)$ with the identities:

$$(a \circ b) \bullet (b \circ c) = b \tag{2.1}$$
$$(a \bullet b) \circ (b \bullet a) = b \tag{2.2}$$

It is relatively easy to show that all semicentral bigroupoids are the composition of an idempotent semicentral bigroupoid and a permutation, that is $a \bullet b = \rho(a \bar\bullet b)$ where $\bar\bullet$ is an idempotent semicentral bigroupoid operation on A and ρ is a permutation of A. It turns out that ρ is the square map of $\bullet$, i.e. $\rho a = a \bullet a$.

In general we can combine any semicentral bigroupoid A with a permutation ρ in this way. We call the resulting semicentral bigroupoid the *lifting* of A by ρ.

Examples of idempotent semicentral bigroupoids include rectangular bands, which are the only associative semicentral bigroupoids.

2

The following is an idempotent nonassociative semicentral bigroupoid of order six.

$\bullet$	1 2 3 4 5 6	$\circ$	1 2 3 4 5 6
1	1 1 3 4 4 3	1	1 2 1 1 2 2
2	2 2 5 6 5 6	2	1 2 1 1 2 2
3	1 1 3 4 4 3	3	3 6 3 3 6 6
4	1 1 3 4 4 3	4	4 5 4 4 5 5
5	2 2 5 6 5 6	5	3 5 3 3 5 5
6	2 2 5 6 5 6	6	4 6 4 4 6 6

We will often omit the $\bullet$ symbol and use juxtaposition to denote the operation where it causes no confusion.

The semicentral bigroupoid axioms are symmetric in the two operations. Thus statements about $(A, \bullet)$ apply to $(A, \circ)$ equally. There is no known equational definition of a semicentral bigroupoid using only one operation. There is a combinatorial description of semicentral bigroupoids which only requires us to investigate one of the operations. As is the case in lattices, if $(A, \bullet, \circ)$ and $(A, \bullet, *)$ are semicentral bigroupoids, then $(A, \circ) = (A, *)$ (not just isomorphic).

Semicentral bigroupoids are *rectangular*, $ab = cd \Rightarrow ad = cb = ab$. Thus for any element $a \in A$ we obtain sets $L_a, R_a \subseteq A$ such that $l \in L_a, r \in R_a \Leftrightarrow lr = a$. For any $a, b \in A$, $|L_a| = |L_b|, |R_a| = |R_b|$ and $|L_a||R_a| = |A|$. This ordered pair $(|L_a|, |R_a|)$ is called the *format* of the semicentral bigroupoid A.

These pairs $\{(L_a, R_a) | a \in A\}$ form a combinatorial structure known as a *rectangular structure*.

Definition 1 *A* Rectangular Structure *on a set S, called the* base set*, is a collection $\mathcal{R}$ of ordered pairs of subsets, called* rectangles*, of S, such that*

$$\forall (s,t) \in S^2 \; \exists! \; R \in \mathcal{R} \; such \; that \; (s,t) \in R \tag{2.3}$$
$$\forall R, Q \in \mathcal{R}, |R_1 \cap Q_2| = 1 \tag{2.4}$$

where we identify $R = (R_1, R_2) = R_1 \times R_2$.

There is a one–to–one correspondence between idempotent semicentral bigroupoids and rectangular structures.

3

There are some special classes of rectangular structures. Two partitions Π, Θ of a set are called *orthogonal* if $\forall P \in \Pi, T \in \Theta, |P \cap T| = 1$.

Definition 2 *A* partitioned rectangular structure *is defined by a set S, a partition Π of S called the* primary partition *and a collection $\{\Theta_\pi : \pi \in \Pi\}$ of partitions of S that are orthogonal to Π.*

The rectangular structure is then defined as $\{(\pi, T) : \pi \in \Pi, T \in \Theta_\pi\}$ for a left–partitioned rectangular structure *and $\{(T, \pi) : \pi \in \Pi, T \in \Theta_\pi\}$ for a* right–partitioned rectangular structure.

It is simple to show that these satisfy the axioms (2.3) and (2.4). They have a simplified structure and special properties. The example of order 6 above is left–partitioned with primary partition 134|256 and secondary partitions 12|36|45 and 12|35|46.

3 Conservation Laws

We are interested in conservation laws, which are numerical properties of states of the RCA that do not change over time, i.e. with applications of the global mapping.

An (additive) conservation law (*conslaw*) is a mapping ϕ from A to the reals such that, if we define $\Phi(a) := \sum \phi(a_i)$ where this sum is defined, then $\Phi(a) = \Phi(Fa)$ where F is the global mapping of the CA.

The set of conslaws for a given CA rule is a vector space over the reals. Thus the problem is to find a basis for the vector space of conslaws for a given rule. The mapping $\phi_r : a \mapsto r$ taking all elements of A to a given real r is a trivial conservation law. Thus we can force one element $o \in A$ to have $\phi(o) = 0$: if $\bar\phi$ is a conslaw, then $\phi(a) = \bar\phi(a) - \bar\phi(o)$ is a conslaw with this property.

Let us consider a CA with only two cells. The conslaw requirement then states that

$$\phi a + \phi b = \phi(ab) + \phi(ba) \; \forall a, b \in A \tag{3.1}$$

We call this the *two–cell requirement*.

Hattori and Takesue [HT91] have demonstrated that the conslaws must satisfy a simple equation for all $x, y \in A$:

$$\phi(x) - \phi(ox) + \phi(oy) - \phi(xy) = 0 \tag{3.2}$$

4

Lemma 3 *Let ϕ be a conslaw for a nonidempotent semicentral bigroupoid $(A, \cdot)$ with square map ρ. Then ϕ will be constant on all orbits of ρ and ϕ will be a conslaw on the idempotent lifting of A. Conversely, if ϕ is a conslaw of an idempotent semicentral bigroupoid $(A, *)$ and ϕ is constant on the orbits of the permutation ρ, then ϕ is a conslaw on the lifting of A by ρ.*

Proof:For the first statement, take $a \in A$, then $\phi a = \phi \rho a$ by (3.1). So ϕ is constant on ρ–orbits.

Let ϕ satisfy (3.2) on $(A, \cdot)$ and be constant on orbits of the permutation ρ. Then $\phi(a * b) = \phi \rho^{-1}(a \cdot b) = \phi(a \cdot b)$ so ϕ satisfies (3.2) on $(A, *)$ and is a conslaw.

Thus the rest of the forward argument and the converse statement can be seen to be true. $\square$

Thus we need only concern ourselves with idempotent semicentral bigroupoids for the rest of this paper.

The equations (3.2) can be formulated as a linear algebra problem. We label the elements of A as $\{1, \ldots, n\}$, then a matrix M is defined so that if $Mv = 0$ then $\phi(i) = v_i$ is a conslaw. The columns are indexed by the elements of A, the rows correspond to the pairs $(x, y) \in A^2$ that give us the equation.

Lemma 4 *The entries of the matrix M are $-1, 0, 1$. The row and column sums are equal to 0.*

Proof:For an entry of M to be 2, we require $x = oy$. Thus $xx = oy$ so by the rectangular property, $xx = xy = ox = oy$ and (3.2) becomes $0 = 0$. Similarly if an entry of M is to be -2, we require $ox = xy$. By the rectangular property, $ox = oy = xx = xy$ so (3.2) reduces to $0 = 0$.

The rows of the matrix thus consist of either all 0s, or exactly one 1 and one -1 or exactly two 1s and two -1s. The row sum is always equal to 0.

In order to calculate the column sum, we count the occurrences of a given element z in pairs x, y. We see that z occurs equally often as ox and oy by symmetry, and equally often ($|A|$ times) as x (the pairs (z, y)) and as xy (the rectangle of pairs (x, y) such that $xy = z$). Thus the sum is zero. $\square$

With these results it has been possible to calculate the space of conslaws for all 1DRCA with A up to order 9, using the exhaustive generation results explained in [Boy03]. Tests have been performed with randomly generated examples of orders 12 and 16. We will return to the results of these tests later.

4 Morphisms and congruences

If $\psi : A \to B$ is a semicentral bigroupoid morphism, and ϕ is a conslaw for B, then $\phi \circ \psi$ is a conslaw for A. We call these *pullbacks* of conservation laws.

Define $K(A)$ as follows: if A is simple, then $K(A)$ is the set of conservation laws of A, otherwise, it is the set of pullbacks of $K(\bar{A})$ for each homomorphic image $\bar{A}$ of A. We obtain the following.

Lemma 5 $K(A)$ *is defined by the maximal congruences, i.e. the simple images of A only.*

Thus, given an arbitrary RCA, we can obtain a class of conservation laws by looking only at the factors modulo the maximal congruences, i.e. the simple images of the algebra.

The question arises as to simple semicentral bigroupoids. The known examples are the left and right constant semicentral bigroupoids of order 2 and two examples of order 9. There is no reason to believe that there are not more. However, the following results show that they do not have a particularly simple structure.

Definition 6 *A rectangular structure $\mathcal{R}$ on a set A is* left (right) partitionable *if there exists a nontrivial partition Π of A such that $\forall R \in \mathcal{R}$, $\forall a \in R_1 : R_1 \subseteq [a]_\Pi$ ($\forall R \in \mathcal{R}$, $\forall a \in R_2 : R_2 \subseteq [a]_\Pi$).*

Lemma 7 *Let $\mathcal{R}$ be a rectangular structure on the set A. If $\mathcal{R}$ is left (right) partitionable then $\mathcal{R}/\Pi$ is a rectangular structure on A/Π with format $1 \times |\Pi|$ ($|\Pi| \times 1$).*

Proof:We concentrate on the left partitionable case. Fix $R \in \mathcal{R}$. It is clear that $|R_1/\Pi| = 1$. For all $a \in A$ s.t. $[a] \in R_2/\Pi$, $\exists Q \in \mathcal{R}$ such that $a \in Q_1$. Then $R_2 \cap Q_1 \neq \emptyset \Rightarrow [a] \in R_2/\Pi$, i.e. $R_2/\Pi = A/\Pi$. Thus we have a $1 \times |\Pi|$ rectangular structure. A similar argument demonstrates the result for right partitionable rectangular structures. $\square$

Define a rectangular structure to be *simple* if the corresponding semicentral bigroupoid is simple.

Corollary 8 *Partitioned semicentral bigroupoids are simple iff of order 2.*

This follows as partitioned implies partitionable.

Corollary 9 *A semicentral bigroupoid of format $2 \times n$ or $n \times 2$ is simple iff $n = 1$.*

Proof:Without loss of generality we consider format $2 \times n$. Suppose $n \neq 1$. The left side of a $2 \times n$ rectangle is a pair. We have $2n$ pairs on $2n$ points in which every point appears in exactly two pairs. Thus we have a graph which is a union of cycles. This is partitionable (and thus not simple) unless there is a unique cycle covering all $2n$ points.

We label the elements of $A = \{1, 2, \ldots, 2n\}$, in order along this cycle. Take some rectangle R. If the unique $a \in R_1 \cap R_2$ is even, then R_2 consists of only even elements, as it must contain n elements and no two may be adjacent on the cycle by the unique intersection property. Similarly if a is odd, then R_2 consists only of odd elements. Thus the rectangular structure is right partitioned, and not simple. □

Thus a simple idempotent semicentral bigroupoid of order higher than 2 must have order at least 3^2, and it turns out that there are precisely two such simple semicentral bigroupoids by investigating the exhaustive lists generated in [Boy03].

The simple semicentral bigroupoids of order 2 have the conslaw that maps one element to 0, the other to 1. The simple examples of order 9 have no nontrivial conslaws.

Lemma 10 *A rectangular structure is partitionable iff the idempotent semicentral bigroupoid can be mapped to a semicentral bigroupoid of order 2.*

Proof:The forward direction is clear, as the partition forms a congruence in the semicentral bigroupoid. We can collapse any partition into a two–class partition to obtain a two element image.

For the reverse direction we use the partition generated by the congruence classes of the homomorphism. Since the image has two elements, wlog the corresponding rectangular structure is of format 1×2. Thus R_1 lies within one class of the partition for all rectangles R. Thus the rectangular structure is partitionable. □

5 0,1 conslaws

All calculated examples to date have a conslaw basis with $\{0, 1\}$–vectors, that is, the vectors contain only entries from the set $\{0, 1\}$. The following results show that this means that all known conslaws are $K(A)$ type.

Lemma 11 *Let $\phi : A \to \{0, 1\}$ be a conslaw. If $\phi a = 0$, $\phi b = 1$ and $\phi(ab) = 0$ then $\phi(ac) = 0$ and $\phi(bc) = 1$ for all $c \in A$.*

Proof:By the Hattori–Takesue equation with $o = a, x = b, y = c$ we have

$$
\begin{align}
0 &= \phi b - \phi(ab) + \phi(ac) - \phi(bc) \tag{5.1} \\
\phi(bc) - \phi(ac) &= 1 \tag{5.2} \\
\phi(bc) &= 1 \text{ and } \phi(ac) = 0 \tag{5.3}
\end{align}
$$

$\square$

Lemma 12 *Let $\phi : A \to \{0, 1\}$ be a conslaw. If $\phi a = 0$, $\phi b = 1$ and $\phi(ab) = 1$ then $\phi(ca) = 0$ and $\phi(cb) = 1$ for all $c \in A$.*

Proof:First note that by the hypothesis, $\phi(ba) = 0$ using the two–cell requirement (3.1). Assume $\phi(ca) = 1$. Then by the two–cell requirement,

$$\phi a + \phi c = \phi c = \phi(ca) + \phi(ac) \geq 1 \Rightarrow \phi c = 1$$

Furthermore

$$2 = \phi b + \phi c = \phi(bc) + \phi(cb) \Rightarrow \phi(bc) = \phi(cb) = 1$$

Then by Hattori and Takesue we obtain

$$0 = \phi c - \phi(bc) + \phi(ba) - \phi(ca) = 1 - 1 + 0 - 1$$

which is a contradiction, thus $\phi(ca) = 0$. From the two–cell requirement

$$\phi a + \phi c = \phi c = \phi(ac) + \phi(ca) = \phi(ac)$$

so the Hattori–Takesue equation

$$\phi c - \phi(ac) + \phi(ab) - \phi(cb) = \phi(ab) - \phi(cb) = 0$$

implies that $\phi(cb) = \phi(ab) = 1$ and we are done. $\square$

Theorem 13 *If $\phi : A \to \{0, 1\}$ is a conslaw, then ϕ is a morphism of A onto a semicentral bigroupoid of order 2.*

Proof:Fix some $a, b \in A$ such that $\phi a = 0$ and $\phi b = 1$. Define an operation $*$ on $\{0, 1\}$ by

$*$	0	1
0	0	$\phi(ab)$
1	$\phi(ba)$	1

We claim that $\phi : (A, \bullet) \to (\{0, 1\}, *)$ is a morphism. By the two–cell requirement, exactly one of $\phi(ab)$ and $\phi(ba)$ is 0 and the other is 1.

Case 1: $\phi(ab) = 0$. In this case $x * y = x$. By Lemma 11 we know that $\phi(ac) = 0$ and $\phi(bc) = 1$ for all $c \in A$. If $\phi c = 1$ then replacing a, b, c with a, c, d in the same Lemma, we see that $\phi(cd) = 1$ for all $d \in A$, so $\phi(cd) = \phi(c) * \phi(d)$ and ϕ is a morphism. If $\phi c = 0$ then by the two–cell requirement $\phi b + \phi c = 1 = \phi(bc) + \phi(cb) \Rightarrow \phi(cb) = 0$. Replacing a, b, c with c, b, d in Lemma 11 we obtain $\phi(cd) = 0$ so $\phi(cd) = \phi c * \phi d$ and ϕ is a morphism.

Case 2: $\phi(ab) = 1$. In this case $x * y = y$. By Lemma 12 we know that $\phi(ca) = 0$ and $\phi(cb) = 1$ for all $c \in A$. If $\phi c = 1$, then $\phi(ac) = \phi a + \phi c - \phi(ca) = 1$ so replacing a, b, c with a, c, d in Lemma 12 implies $\phi(dc) = 1 = \phi d * \phi c$, so ϕ is a morphism. If $\phi c = 0$ then replacing a, b, c with c, b, d in Lemma 12 gives us $\phi(dc) = 0 = \phi d * \phi c$ and ϕ is a morphism. $\qquad\square$

Thus if we can find a conslaw space with no $\{0, 1\}$–basis, then we have something special. Otherwise we find:

Conjecture 14 *There is a one-to-one correspondence between the basis of the space of nontrivial conservation laws of a CA rule and the maximal congruences of A with factor of size 2.*

This result is similar to the result from classical continuous dynamical systems that connects the space of automorphisms of the system with its conserved quantities. Note that examples show that a similar result doesn't apply for the group of automorphisms of the algebra. This was my starting point and it doesn't work except in the trivial cases.

Note that if this conjecture is true, we obtain that the following concepts are equivalent:

- Additive conservation laws in one dimensional reversible cellular automata
- Maximal congruences on semicentral bigroupoids with two element factors
- Partitionability of rectangular structures.

This three–way connection seems a little strange. It would imply that the dimension of the conslaw vector space is constant, independent of which field we take for it.

6 Conclusion

We investigated the properties of additive conservation laws in one dimensional reversible cellular automata with a binary local rule. We have shown that a class of such conservation laws exist, determined by the maximal congruences of index two in the groupoid defined by the local rule. Exhaustive testing of examples has shown that only such laws exist. We demonstrated that these conservation laws correspond exactly to those with a 0,1 basis.

We conjecture that this holds in general. Two results would be of interest. Showing that no nontrivial conservation laws exist on simple semicentral bigroupoids of order greater than two would strengthen but not prove the conjecture. More important would be to determine that all nullspaces of the conservation law defining matrix have a 0,1 basis. In fact, limited experiments in general binary local rule cellular automata indicate that this might be a general result: all conservation laws lie in a space with a basis determined by the maximal congruences of order two in the groupoid.

7 Acknowledgments

The work here started during a visit to Cris Moore at the Sante Fe institute. After working with him for some time, we realised that Hattori and Takesue had obtained all our results years before!

References

[Boy03] Tim Boykett. Efficient exhaustive listings of reversible one dimensional cellular automata. Accepted, in press, 2003.

[HT91] Tetsuya Hattori and Shinji Takesue. Additive conserved quantities in discrete-time lattice dynamical systems. *Physica D*, 49:295–322, 1991.

[Ped92] John Pedersen. Cellular automata as algebraic systems. *Complex Systems*, 6:237–250, 1992.

[Ric72] D. Richardson. Tesselations with local transformations. *J. Comp. Sys. Sci.*, 6:373–388, 1972.